Preprint typeset in JHEP style - HYPER VERSION

Scaling laws near the conformal window of many-flavor QCD

Jens Braun and Holger Gies

Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena,
Max-Wien-Platz 1, D-07743 Jena, Germany

Abstract: We derive universal scaling laws for physical observables such as the critical temperature, the chiral condensate, and the pion decay constant as a function of the flavor number near the conformal window of many-flavor QCD in the chiral limit. We argue on general grounds that the associated critical exponents are all interrelated and can be determined from the critical exponent of the running gauge coupling at the Caswell-Banks-Zaks infrared fixed point. We illustrate our findings with the aid of nonperturbative functional Renormalization Group (RG) calculations and low-energy QCD models.

Contents

1. Introduction

Many-flavor QCD has recently attracted a great deal of attention for a number of reasons: first, as a controllable deformation of real QCD, it can teach important lessons about the chiral structure of QCD-like theories. Second, it serves as a building block for alternative technicolor-like scenarios for the Higgs sector. And third, it exhibits a quantum phase transition from the QCD-like chirally broken to a conformal phase as a function of the flavor number and thus gives rise to interesting quantum critical behavior.

Adding $N_{\rm f}$ massless quark flavors to a nonabelian $\mathrm{SU}(N_{\rm c})$ gauge theory increases the screening property of fermionic fluctuations. An obvious consequence is the loss of asymptotic freedom for $N_{\rm f} > N_{\rm f}^{\rm a.f.} := \frac{11}{2} N_{\rm c}$ ($= 16.5$ for SU(3)). Already at smaller $N_{\rm f}$, $N_{\rm f} > \frac{34 N_{\rm c}^3}{13 N_{\rm c}^2 - 3}$ ($\simeq 8.05$ for SU(3)), the second β function coefficient changes sign inducing an infrared (IR) attractive fixed point of the gauge coupling $\alpha_* > 0$ [1]. For small $N_{\rm f}^{\rm a.f.} - N_{\rm f} > 0$, the fixed-point value is small. This gives access to a perturbative analysis, suggesting that the system approaches a conformally invariant limit in the deep IR [2]. For smaller $N_{\rm f}$ (such as real QCD), this Caswell-Banks-Zaks fixed point is destabilized due to the spontaneous break-down of chiral symmetry, resulting in massive fermionic excitations, strongly-coupled glue and massless Goldstone bosons.

The above considerations propose the existence of a critical flavor number $N_{\rm f}^{\rm cr}$, separating the chiral-symmetry-broken phase from the conformal phase. Theories with a flavor number $N_{\rm f}$ satisfying $N_{\rm f}^{\rm cr} \leq N_{\rm f} < N_{\rm f}^{\rm a.f.}$ are said to be in the conformal window. The flavor number $N_{\rm f}$ therefore serves as a control parameter for a quantum phase transition. Investigations of this phase structure have been performed by continuum methods [3, 4, 5, 6, 7, 8, 10, 9, 11, 12, 13, 14, 15, 16, 17], as well as lattice simulations [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32]. Recent results have collected a

substantial body of evidence for the existence of the conformal phase and a critical flavor number, defining the onset of the conformal phase: lattice simulations have provided evidence that $8 < N_{\rm f}^{\rm cr} \leq 12$ [25, 26, 27, 28, 29], even though the case $N_{\rm f} = 12$ is controversial, see [31]. On the present level of accuracy, these results go well together with an earlier quantitative estimate from a combination of four-loop perturbation theory with the functional Renormalization Group (RG) which yields $N_{\rm f}^{\rm cr} = 10.0^{+1.6}_{-0.7}$ [12].

Beyond the precise location of this quantum critical point on the $N_{\rm f}$ axis, the critical behavior in the vicinity of the fixed point is expected to show several peculiarities: for $N_{\rm f} < N_{\rm f}^{\rm cr}$ in the chirally broken phase, a standard chiral phase transition can be anticipated to occur at finite temperature. The critical phenomena near this finite-T chiral phase transition are expected to be determined by the precise symmetries of the order parameter, i.e., the chiral condensate, defining the universality class of the transition (e.g., $\mathrm{SU}(2)_{\rm L}\otimes \mathrm{SU}(2)_{\rm R} \simeq \mathrm{O}(4)$ for $N_{\rm f} = 2$) [33]. Near the critical temperature an effective description in terms of a Ginzburg-Landau-type effective potential is expected to hold even on a quantitative level. By contrast, the zero-temperature quantum phase transition as a function of the control parameter $N_{\rm f}$ may not have a continuous Ginzburg-Landau description. In particular, there appear to be no light scalar states in terms of which an effective theory could be constructed on the conformal side $N_{\rm f} \gtrsim N_{\rm f}^{\rm cr}$ of the phase transition [3, 4, 34], even though the order parameter (chiral condensate) should change continuously across the phase transition.

Plotting the chiral-phase-transition temperature versus the control parameter $N_{\rm f}$ yields the phase boundary in the $(T, N_{\rm f})$ plane which has first been explored in [13, 14] using the functional RG. Near the critical flavor number, an intriguing relation between the shape of the phase boundary and RG properties of the Caswell-Banks-Zaks fixed point has been identified. This relation connects the scaling of the critical temperature with the critical exponent of the gauge coupling near the Caswell-Banks-Zaks fixed point. In the present work, we argue that this scaling relation can be extended to further physical observables such as the chiral condensate or the pion decay constant. As these observables are accessible to a variety of nonperturbative methods, our results suggest that the corresponding scaling can become a useful tool to study the phase transition to the conformal phase quantitatively.

Our main arguments are based on very general considerations and involve only few assumptions about the RG structure of the theory. These are illustrated in Sect. 2 for the simple few-flavor case and in Sect. 3 for many flavors near the conformal window. These arguments are then made more concrete with the aid of functional RG calculations in a derivative expansion of the QCD effective action, or simply within low-energy QCD models in Sect. 4. Throughout this work, we concentrate on the chiral phase transition even though we also expect an impact of the confining nature of the theory on the properties of the system near criticality. However, as we work in the chiral limit, there is no good order parameter for confinement, implying that nonanalyticities in the correlations are rather dominated by the chiral degrees of freedom.

2. Few-flavor QCD and the role of scale fixing

QCD in the chiral limit of zero quark masses depends only on one parameter in the Euclidean Lagrangian, namely the gauge coupling g,

$$\mathcal{L} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + i\bar{\psi}\gamma_\mu D_\mu[A]\psi. \tag{2.1}$$

In the quantum theory, the gauge coupling has to be fixed at a certain momentum scale in terms of a renormalization condition. The renormalization group finally trades the gauge coupling fixed at an arbitrary scale for one single parameter $\Lambda_{\rm QCD}$ of mass dimension one which sets the mass scale for all physical observables of the theory. In other words, all physical observables respond trivially to a variation of $\Lambda_{\rm QCD}$ according to their engineering dimension. In units of $\Lambda_{\rm QCD}$, the theory is completely fixed.

In order to discuss the dependence on quantities such as the flavor number, it is important to emphasize that a variation of the flavor number does not merely correspond to a change of a parameter of the theory. It rather corresponds to changing the theory itself. In particular, there is no unique way to unambiguously compare theories of different flavor number with each other, as different theories may have different scales $\Lambda_{\rm QCD}$.

For instance, it might seem natural to compare theories with different flavor numbers at fixed $\Lambda_{\rm QCD}$ with each other. But $\Lambda_{\rm QCD}$ itself is not a direct observable, so that such a comparison is generically inflicted with theoretical uncertainties. Moreover, $\Lambda_{\rm QCD}$ is regularization-scheme dependent which can affect comparisons between different theoretical methods, say, lattice and continuum results. Another option could be a scale fixing in the deep perturbative region, say, at the Z mass pole by fixing $\alpha(M_Z)$. However, theories with different flavor numbers then exhibit a different perturbative running, such that IR observables vary because of both high-scale perturbative as well as non-perturbative evolution.

Instead, we propose to choose a mid-momentum scale for the scale fixing, as the high-scale perturbative running is then separated from the more interesting non-perturbative dynamics. In this work, we fix the theories at any $N_{\rm f}$ by keeping the running coupling at the τ mass scale fixed to $\alpha(m_\tau) = 0.322$. Even though also this choice is scheme dependent, these dependencies should be subdominant, as they follow a perturbative ordering. In general, fixing the scale via the coupling is a prescription which is well accessible by many nonperturbative methods.

Let us now present a simple argument that illustrates how $N_{\rm f}$ dependencies of physical observables can be understood in the limit of small $N_{\rm f}$. As already stated above, all IR observables such as the critical temperature $T_{\rm cr}$, pion decay constant f_π, chiral condensate $\langle\bar{\psi}\psi\rangle^{1/3}$, and model-dependent concepts such as the constituent quark mass, are proportional to $\Lambda_{\rm QCD}$. The latter on the one hand can be read off from the UV behavior of the running coupling, $\alpha(k) \sim 1/\ln(k/\Lambda_{\rm QCD})$ for large k. On the other hand, the value of $\Lambda_{\rm QCD}$ can be associated with the position of the Landau pole in perturbation theory.[1] In

[1] Of course, this statement has to be taken with care, since $\Lambda_{\rm QCD}$ is a meaningful scale, whereas the Landau pole is simply an artifact of perturbation theory.

this simple sense, the artificial Landau pole can be taken as an estimate for the scaling of physical observables. In one-loop RG-improved perturbation theory, the position of the Landau pole can be read off from

$$0 \leftarrow \frac{1}{\alpha(\Lambda_{\mathrm{QCD}})} = \frac{1}{\alpha(\mu_0)} + 4\pi b_0 \,\ln \frac{\Lambda_{\mathrm{QCD}}}{\mu_0},$$
$$b_0 = \frac{1}{8\pi^2}\left(\frac{11}{3}N_{\mathrm{c}} - \frac{2}{3}N_{\mathrm{f}}\right), \tag{2.2}$$

where μ_0 denotes a perturbative scale, such as $m_\tau, M_Z, \ldots$. Solving this equation for Λ_{QCD} and expanding the result for small N_{f} leads us to

$$\Lambda_{\mathrm{QCD}} \simeq \mu_0 \,\mathrm{e}^{-\frac{1}{4\pi b_0 \alpha(\mu_0)}}$$
$$\simeq \mu_0 \,\mathrm{e}^{-\frac{6\pi}{11 N_{\mathrm{c}} \alpha(\mu_0)}} \left(1 - \epsilon N_{\mathrm{f}} + \mathcal{O}((\epsilon N_{\mathrm{f}})^2)\right).$$

Choosing $\mu_0 = m_\tau$, we find $\epsilon = \frac{12\pi}{121 N_{\mathrm{c}}^2 \alpha(\mu_0)} \simeq 0.107$ for $N_{\mathrm{c}} = 3$. Two conclusions can immediately be drawn: first, Λ_{QCD} can be expanded in N_{f} and has a generically nonvanishing linear term; second, for the present way of scale fixing, the linear behavior should be a reasonable approximation for finite values of N_{f}, say $N_{\mathrm{f}} \lesssim 4$, as the expansion parameter ϵ is small.

As Λ_{QCD} sets the scale for all dimensionful IR observables, we are tempted to conclude that all IR observables scale linearly with N_{f} for small N_{f} with the same proportionality constant ϵ. This is, of course, a bit too simplistic, as the dynamics which establishes the value of the IR observables generically carries an N_{f} dependence as well. E.g., the chiral symmetry-breaking dynamics depends on the number of light mesonic degrees of freedom, which is an N_{f}-dependent quantity. Detailed quantitative model studies [35], however, demonstrate that the scaling of the critical temperature does not receive strong corrections of this type and indeed scales according to

$$T_{\mathrm{cr}} = T_0(1 - \epsilon N_{\mathrm{f}} + \mathcal{O}((\epsilon N_{\mathrm{f}})^2)), \tag{2.3}$$

where T_0 is a dimensionful proportionality constant. We conclude that the phase boundary in the (T, N_{f}) plane has a linear shape for small N_{f} which can mainly be understood as a result of the perturbative N_{f} scaling of Λ_{QCD}. Note that this observation is consistent with lattice simulations [36] and has been exploited for parameter fixing in PNJL/PQM-model studies [37]. In the following, we will show that the shape of the phase boundary as well as the N_{f} scaling of other observables for large N_{f} can also be understood from simple scaling arguments which this time follow from general properties of the nonperturbative domain.

3. Scaling in many-flavor QCD near the conformal window

In this section, we review, detail and extend the scaling arguments presented in [13, 14], leading to universal relations near the conformal window. Whereas the N_{f} scaling in the few-flavor case essentially follows from analyticity of the observables in N_{f} also for N_{f} near zero, the N_{f} dependence near the conformal window is clearly nonperturbative in N_{f}. The

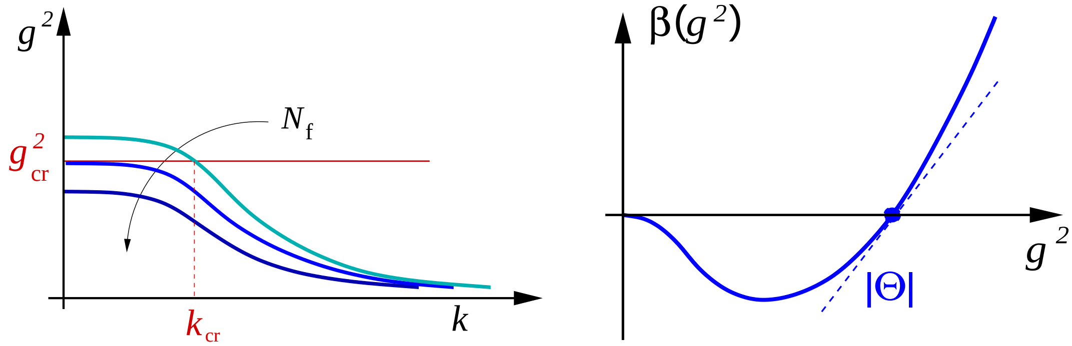

Figure 1: Left panel: illustration of the IR running of the gauge coupling in comparison with the critical coupling g^2_{cr} for χSB. Below the conformal window, $N_f < N_f^{cr}$, g^2 exceeds the critical value, triggering the approach to χSB. For increasing flavor number, the IR fixed-point value g^2_* drops below the critical value denoting the onset of the conformal window. Right panel: sketch of the β function of the running coupling. The slope of the β function at the IR fixed point corresponds to minus the critical exponent Θ, cf. Eq. (3.1).

mere existence of the conformal window requires a sufficient amount of fermionic screening, i.e., a sufficient amount of fermionic degrees of freedom $N_f \geq N_f^{cr}$.

The lower end of the conformal window is characterized by the onset of chiral symmetry breaking. Whereas the coupling approaches the Caswell-Banks-Zaks fixed point g^2_* in the conformal window, chiral symmetry breaking destabilizes this fixed point below the critical flavor number N_f^{cr}. This suggests the existence of a critical coupling g^2_{cr}. If $g^2 > g^2_{cr}$ at some scale, the system will be triggered to run into the chirally broken regime.

For a monotonous coupling flow, the value of the Caswell-Banks-Zaks fixed point g^2_* or its nonperturbative variant corresponds to the maximum possible coupling strength of the system in the conformal window.[2] As both g^2_* and g^2_{cr} depend on the number of flavors, the condition $g^2_*(N_f^{cr}) = g^2_{cr}(N_f^{cr})$ defines the lower end of the conformal window and thus the critical flavor number, see left panel of Fig. 1 for an illustration. For $g^2_* > g^2_{cr}$, the model is below the conformal window and runs into the broken phase. Slightly below the conformal window, the running coupling g^2 exceeds the critical value while it is in the attractive domain of the IR fixed point g^2_*. The flow in this fixed-point regime can approximately be described by the linearized β function

$$\beta_{g^2} \equiv \partial_t g^2 = -\Theta\,(g^2 - g^2_*) + \mathcal{O}((g^2 - g^2_*)^2), \tag{3.1}$$

[2]The following scenario does not apply if the running coupling overshoots, develops a local maximum and approaches the fixed point from above which requires a double-valued β function. This behavior is seen, e.g., in MOM-scheme running couplings derived from the ghost-gluon vertex using truncated Dyson-Schwinger equations [38] for pure Yang-Mills theory. If such a scenario held in many-flavor QCD, the occurrence of χSB, i.e., whether or not the fermion sector becomes critical for a given N_f, would depend quantitatively on the details of the coupling flow, making it difficult to extract universal features. However, we do not expect such a running to occur in many-flavor QCD near the conformal window as the fast running of fermions in the near-critical region reduces the fermionic screening contributions, thus supporting a monotonous increase of the gauge coupling. Also, such a behavior is not observed near the upper end of the conformal window where perturbation theory is expected to hold.

where $t = \ln(k/\Lambda)$ with k being a suitable RG scale and Λ defining the UV cutoff. The universal "critical exponent" Θ denotes (minus) the first expansion coefficient. We know that $\Theta < 0$, since the fixed point is IR attractive, see right panel of Fig. 1. In general, the critical exponent depends on N_f, $\Theta = \Theta(N_\mathrm{f})$. The solution to Eq. (3.1) for the running coupling in the fixed-point regime reads

$$g^2(k) = g_*^2 - \left(\frac{k}{k_0}\right)^{-\Theta}, \tag{3.2}$$

where the scale k_0 is implicitly defined by a suitable initial condition and is kept fixed in the following. It corresponds to a scale where the system is already in the fixed-point regime, and otherwise plays the same role as the renormalization scale μ_0 in Eq. (2.2); in particular, physical observables are independent of k_0. For the present fixed-point considerations, it provides for all dimensionful scales in the following. But knowing the full RG trajectory, k_0 is related to μ_0 and thus, say, to the initial τ mass scale by RG evolution.

Our criterion for χSB is that $g^2(k)$ should exceed g_cr^2 for some value of $k \leq k_\mathrm{cr}$. From Eq. (3.2) and the condition $g^2(k_\mathrm{cr}) = g_\mathrm{cr}^2$, we derive the estimate valid in the fixed-point regime

$$k_\mathrm{cr} \simeq k_0\,(g_*^2 - g_\mathrm{cr}^2)^{-\frac{1}{\Theta}}. \tag{3.3}$$

This scale k_cr now takes over the role of the fixed renormalization scale $\mu_0 = m_\tau$ in the small-N_f argument given above: it sets the scale for the critical temperature $T_\mathrm{cr} \sim k_\mathrm{cr}$ with a proportionality coefficient provided by the solution of the full flow. The last step of the argument goes along with the estimate that the IR fixed-point value g_*^2 roughly depends linearly on N_f. More precisely, we assume that the N_f dependence of the coupling quantities can be linearized near the critical flavor number. From Eq. (3.3), we thus find the relation

$$T_\mathrm{cr} \sim k_0 |N_\mathrm{f} - N_\mathrm{f}^\mathrm{cr}|^{-\frac{1}{\Theta}}, \tag{3.4}$$

which is expected to hold near N_f^cr for $N_\mathrm{f} \leq N_\mathrm{f}^\mathrm{cr}$. Here, Θ should be evaluated at N_f^cr.[3] Relation (3.4) is an analytic prediction for the shape of the chiral phase boundary in the (T, N_f) plane of QCD. This result is remarkable for a number of reasons: first, it relates two universal quantities with each other: the phase boundary and the IR critical exponent. Second, it establishes a quantitative connection between the chiral structure (T_cr) and the IR gauge dynamics (Θ). Third, it is a parameter-free prediction following essentially from scaling arguments.

As Eq. (3.4) relates two universal quantities, it is important to understand to what extent the underlying argument makes use of non-universal but scheme-dependent quantities. Of course, the running coupling is a strongly scheme-dependent concept, and so is the value of the IR fixed point g_*^2. However, the existence of the fixed point as well as the value of the critical exponent are scheme independent.[4] Also, the value of the critical coupling

[3]Accounting for the N_f dependence of Θ by an expansion around N_f^cr yields mild logarithmic corrections to Eq. (3.4).

[4]The running coupling in addition is definition dependent. Here, we assume that the running coupling used in the discussion provides for a reasonable measure of the interaction between the gauge and the quark sector which manifestly exhibits the IR fixed point in the conformal window.

$g_{\rm cr}^2$ is scheme dependent. Nevertheless this scheme dependence has to cancel against that of the running coupling itself, as whether or not χSB occurs is a universal feature of the system.

Furthermore, we have implicitly neglected all dependencies of the running on scales other than the RG scale k. At finite temperature, there will, of course, be dependencies of the coupling on T in particular at low scales where T/k becomes large. The relevant scale for the above argument, however, is the scale $k_{\rm cr}$ where χSB is triggered. We expect that this scale is generically somewhat larger than the temperature as long as we are in the χSB phase or approach the phase transition from below. Indeed, this assumption turns out to hold in all model calculations as well as in the RG results described below. This allows us to ignore the T dependence of the running coupling g^2 and of the critical coupling $g_{\rm cr}$.

Let us now generalize these considerations to other physical observables such as the chiral condensate or the pion decay constant. To be more specific, we are interested in the scaling behavior of physical observables as a function of the number of massless quark flavors $N_{\rm f}$ at vanishing temperature. In this case, the above argument can be followed straightforwardly, where the determination of the critical (RG) scale $k_{\rm cr}$ in Eq. (3.3) plays a prominent role. In terms of low-energy effective theories, the scale $k_{\rm cr}$ can be viewed as an ultraviolet (UV) cutoff. From Eqs. (3.3) and (3.4), it follows immediately that the critical scale and therewith this UV cutoff $\Lambda_{\rm eff}$ of the low-energy sector is tightly related to the critical flavor number:

$$\Lambda_{\rm eff} \sim k_{\rm cr} \simeq k_0\,|N_{\rm f} - N_{\rm f}^{\rm cr}|^{-\frac{1}{\Theta}}. \tag{3.5}$$

On the other hand, observables $\mathcal{O}$ with mass dimension $d_{\mathcal{O}}$ which are computable in this effective field theory defined by the fixed-point regime are necessarily related to the UV cutoff $\Lambda_{\rm eff}$ in a simple manner,

$$\mathcal{O} \simeq c_{\mathcal{O}} \Lambda_{\rm eff}^{d_{\mathcal{O}}}\,, \tag{3.6}$$

where $c_{\mathcal{O}}$ is a numerical constant which depends on the details of the theory, e.g., the number of colors $N_{\rm c}$ and also the number of flavors $N_{\rm f}$. Here, we have assumed that a general separation of scales holds in the sense that all effective UV parameters of the low-energy effective theory are fully determined by the quark-gluon dynamics in the mid- and high-momentum regime. Combining Eqs. (3.5) and (3.6), we find

$$\mathcal{O} \simeq k_0^{d_{\mathcal{O}}} |N_{\rm f} - N_{\rm f}^{\rm cr}|^{-\frac{d_{\mathcal{O}}}{\Theta}}. \tag{3.7}$$

This relation extends the scaling properties of the critical temperature near the conformal window found above in Eq. (3.4) to that of other physical IR observables. Again the universal scaling of these observables as a function of $N_{\rm f}$ is related to the critical exponent Θ of the running coupling.

4. Functional RG results

The critical temperature as a function of $N_{\rm f}$ has first been computed in [13, 14] in the framework of the functional RG [39] (see [40] for reviews on the functional RG in gauge

theories). In this section, we briefly review these results and discuss them in the light of the scaling relations. Result on further IR observables will be discussed in the next section.

In [13, 14], the RG flow of QCD starting from the microscopic degrees of freedom in terms of quarks and gluons was studied within a covariant derivative expansion, approaching the critical χSB temperature from above. A crucial ingredient for χSB are the scale-dependent gluon-induced quark self-interactions of the type

$$\Gamma_{\psi,\mathrm{int}} = \int \hat{\lambda}_{\alpha\beta\gamma\delta} \bar{\psi}_\alpha \psi_\beta \bar{\psi}_\gamma \psi_\delta, \tag{4.1}$$

where $\alpha, \beta, \dots$ denote collective indices including color, flavor, and Dirac structures. These four-fermion interactions are set to zero at the initial UV scale, $\hat{\lambda}_{\alpha\beta\gamma\delta}|_{k\to\Lambda} \to 0$. This guarantees that the $\hat{\lambda}$'s at $k < \Lambda$ are solely generated by quark-gluon dynamics from first principles (e.g., by 1PI "box" diagrams with 2-gluon exchange). This is an important difference to models such as the Nambu-Jona-Lasinio (NJL) model, where the $\hat{\lambda}$'s are independent input parameters.

As we approach the chiral phase transition temperature from above, the derivative expansion with local "point-like" interactions is a self-consistent approximation. This corresponds to replacing the momentum structure of the $\hat{\lambda}$'s by the overall k dependence on the RG scale. This approximation has been successfully tested quantitatively in [12] by verifying the resulting insensitivity of the many-flavor quantum phase transition on the momentum regularization. In the chirally broken regime, this approximation breaks down as, e.g., mesons manifest themselves as momentum singularities in these vertices. These restrictions result in a total number of four linearly independent couplings $\hat{\lambda}_i$ [41]. Introducing the dimensionless couplings $\lambda_i = k^2 \hat{\lambda}_i$, the corresponding β functions read

$$\partial_t \lambda_i = 2\lambda_i - b_{ij}\lambda_j g^2 - A_{ijk}\lambda_j\lambda_k - c_i g^4, \tag{4.2}$$

where the coefficients A, b, c depend on the temperature, number of quark flavors N_f and number of colors N_c; for explicit representations, see [12, 13, 14].

Within this truncation, a simple picture for the chiral dynamics arises, see Fig. 2: at weak gauge coupling, the RG flow generates quark self-interactions of order $\lambda \sim g^4$ via the last term in Eq. (4.2) with a negligible back-reaction on the gluonic RG flow. If the gauge coupling in the IR remains smaller than a critical value $g < g_{\mathrm{cr}}$, the λ self-interactions remain bounded, approaching fixed points λ_* in the IR. The fixed points correspond to a shifted Gaußian fixed point $\lambda_*^{\mathrm{Gauß}}|_{g^2=0} = 0$. At these fixed points, the fermionic subsystem remains in the chirally invariant phase which is indeed realized at high temperatures $T > T_{\mathrm{cr}}$.

The evaluation of the QCD RG flow in a covariant derivative expansion [42, 43] includes further gauge-field operators as well as kinetic terms for the fermion. From the gauge dynamics, the running gauge coupling can be extracted including its dependence on temperature T as well as flavor and color numbers. Following Ref. [13, 14], the increase of the running coupling in the IR is weakened on average for both larger T and larger N_{f}, in agreement with general expectations. In addition, also g_{cr} depends on T and N_{f}, even though the N_{f} dependence is rather weak. For instance, the (non-universal) zero-temperature value of

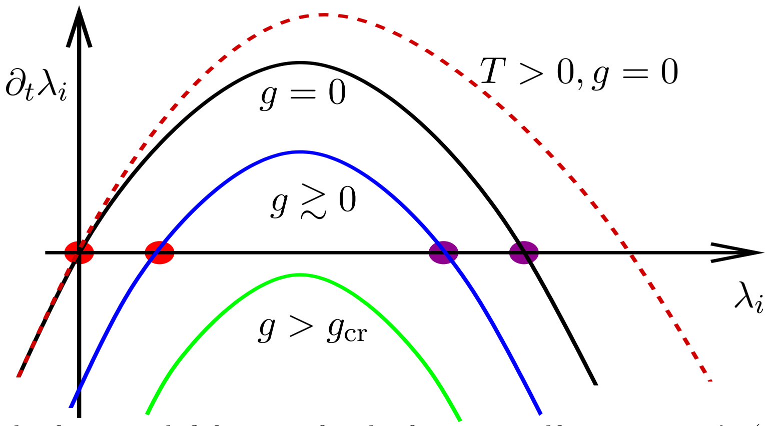

Figure 2: Sketch of a typical β function for the fermionic self-interactions λ_i (taken from [14]): at zero gauge coupling, $g = 0$ (upper solid curve), the Gaußian fixed point $\lambda_i = 0$ is IR attractive. For small $g \gtrsim 0$ (middle/blue solid curve), the fixed-point positions are shifted by the gauge-field fluctuations $\sim g^4$. For gauge couplings larger than the critical coupling $g > g_{\rm cr}$ (lower/green solid curve), no fixed points remain and the self-interactions quickly grow large, signaling χSB. For increasing temperature, the parabolas become broader and higher, owing to thermal fermion masses; this is indicated by the dashed/red line.

the critical coupling for an optimized RG scheme is $\alpha_{\rm cr} = g_{\rm cr}^2/(4\pi) \simeq 0.8$ for $N_{\rm c} = 3$ and a wide range of $N_{\rm f}$ [12].

The T dependence of $g_{\rm cr}$ arises from the quark modes acquiring thermal masses. This leads to a quark decoupling, requiring stronger interactions for critical quark dynamics. In the β function picture of Fig. 2, the λ_i parabolas become broader with a higher maximum; hence, the annihilation of the Gaußian fixed point by pushing the parabola below the λ_i axis requires a larger $g_{\rm cr}$.

At zero temperature and for small $N_{\rm f}$, the IR fixed point g_*^2 is far larger than $g_{\rm cr}^2$, hence QCD is in the χSB phase. For increasing T, the temperature dependence of the coupling and that of $g_{\rm cr}^2$ compete with each other. For the case of many massless quark flavors $N_{\rm f}$, the critical temperature is plotted in Fig. 3. For the scale fixing at the τ mass scale discussed above, we observe an almost linear decrease of the critical temperature for increasing $N_{\rm f}$ with a slope of $\Delta T_{\rm cr} = T(N_{\rm f}) - T(N_{\rm f}+1) \approx 25\,\mathrm{MeV}$ at small $N_{\rm f}$. This linear dependence of the full result confirms the simple estimate given in Eq. (2.3) to a very good accuracy. Moreover, the predicted relative difference for $T_{\rm cr}$ for $N_{\rm f} = 2$ and 3 flavors of $\Delta \simeq 0.146$ is in very good agreement also with lattice studies [36]. We conclude that the shape of the phase boundary for small $N_{\rm f}$ is basically dominated by fermionic screening.

For larger flavor number, the critical temperature decreases and the phase transition line terminates at the zero-temperature quantum phase transition at $N_{\rm f}^{\rm cr}$, denoting the onset of the conformal window. In the RG study of [13, 14], we find a critical number of quark flavors, $N_{\rm f}^{\rm cr} \simeq 12.9$. This result for $N_{\rm f}^{\rm cr}$ agrees with other studies based on the 2-loop β function [2]. However, the precise value of $N_{\rm f}^{\rm cr}$ has to be taken with care: for instance, in a perturbative framework, $N_{\rm f}^{\rm cr}$ turns out to be sensitive to the 3-loop coefficient which is not reliably reproduced in this leading-order study. This coefficient can bring $N_{\rm f}^{\rm cr}$ down

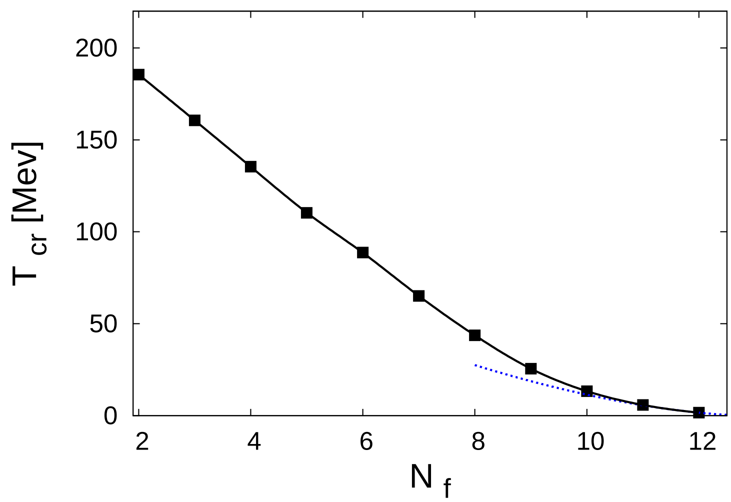

Figure 3: Chiral-phase-transition temperature $T_{\rm cr}$ versus the number of massless quark flavors $N_{\rm f}$ for $N_{\rm f} \geq 2$, as obtained in Ref. [14]. The flattening at $N_{\rm f} \gtrsim 10$ is a consequence of the IR fixed-point structure. The dotted line depicts the analytic estimate near $N_{\rm f}^{\rm cr}$ which follows from the fixed-point scenario (cf. Eq. (3.4)).

to $N_{\rm f}^{\rm cr} \simeq 10.0^{+1.6}_{-0.7}$ which remains stable under the inclusion of 4-loop corrections [12]. The error bars parameterize truncation errors which habe been quantified by artificial scheme dependencies.

To the left of the conformal window ($N_{\rm f} < N_{\rm f}^{\rm cr}$), the phase transition line shows a characteristic flattening. This is again in perfect agreement with our scaling relation (3.4). The fit to numerical results from a functional RG approach is depicted by the dotted line in Fig. 3. In particular, the fact that $|\Theta| < 1$ near $N_{\rm f}^{\rm cr}$ explains the flattening of the phase boundary near the critical flavor number. Within the covariant derivative expansion of RG flow, the IR critical exponent of the β function of the coupling at the critical flavor number yields $\Theta(N_{\rm f}^{\rm cr}) \simeq -0.60$ [14]. However, this estimate is likely to be affected by truncation errors as $\Theta(N_{\rm f})$ is expected to show sizable dependencies on $N_{\rm f}$; hence, any error in $N_{\rm f}^{\rm cr}$ translates into a corresponding error in $\Theta(N_{\rm f}^{\rm cr})$.

For comparison, we plot the perturbative estimates for the critical exponent $\Theta(N_{\rm f})$ in Fig. 4 based on the 4-loop β function in the $\overline{\rm MS}$ scheme [44]. The apparent convergence of the perturbative expansion is remarkable as the difference between the 3- and 4-loop result is below the 1% level (for $8.5 < N_{\rm f} < 16.5$). The 2-loop result shows larger deviations for smaller $N_{\rm f}$, as the fixed-point coupling g_*^2 is larger in this regime. In Fig. 4, we also show the estimate of Θ from the covariant derivative expansion of the RG flow [14]. While this estimate includes nonperturbative contributions to all-loop orders in the gauge sector, the derivative expansion in the fermion sector effectively corresponds to the inclusion of just the (RG-improved) one-loop quark diagram. This explains a large part of the difference to the perturbative estimates. Incidentally, results from further studies of the full β function can be used to estimate Θ. For instance, the Θ values obtained from the conjectured "NSVZ-inspired" β function [45] are identical to the 2-loop result.

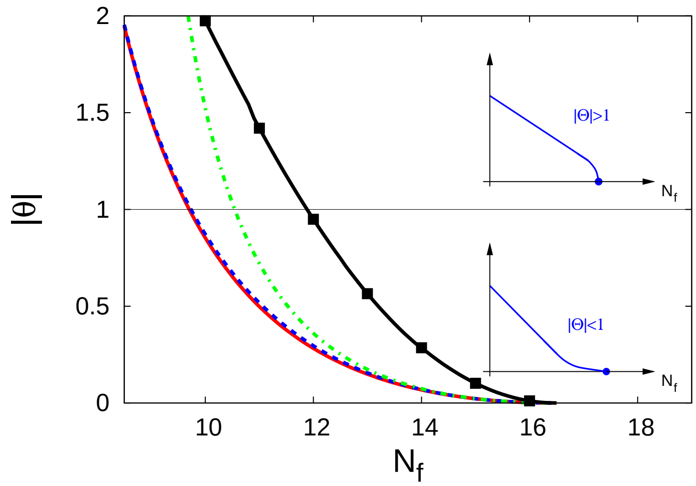

Figure 4: Critical exponent $|\Theta|$ of the running gauge coupling at the Caswell-Banks-Zaks fixed point as a function of the flavor number N_{f}. The perturbative expansion appears to converge, as the difference between the 3-loop (blue/dashed line) and 4-loop result (red/solid line) is below the 1% level (for $8.5 < N_{\mathrm{f}} < 16.5$). The 2-loop result (green/dot-dashed line) shows larger deviations for smaller N_{f}. The black/solid line with symbols corresponds to the estimate from the covariant derivative expansion of the RG flow [14]. The inlays depict the characteristic shapes of the scaling of a generic IR observable $\mathcal{O}$ for $|\Theta| > 1$ (infinite slope at $N_{\mathrm{f}}^{\mathrm{cr}}$) and $|\Theta| < 1$ (vanishing slope at $N_{\mathrm{f}}^{\mathrm{cr}}$).

Qualitatively, the scaling of physical observables near the conformal window shows a characteristic difference for $|\Theta|$ being larger or smaller than one. If $|\Theta| > 1$ the value of a given observable as a function of the distance to the conformal window, $\mathcal{O} = \mathcal{O}(N_{\mathrm{f}}^{\mathrm{cr}} - N_{\mathrm{f}})$, approaches the N_{f} axis with infinite slope. For $|\Theta| < 1$ the slope vanishes on the quantum critical point $N_{\mathrm{f}} = N_{\mathrm{f}}^{\mathrm{cr}}$. This characteristic shape dependence of the scaling is again a firm prediction of our scenario and may be observable in lattice simulations even away from the critical flavor number.

Let us come back to the issue of choosing a specific fixing scale for comparing theories with different N_{f} to each other. As stressed above, e.g., the result for the shape of the phase boundary in Fig. 3 does depend on our choice of fixing the running coupling at the τ-mass scale m_τ. This choice is not unique: in principle, the fixing scale can be chosen as a free function of N_{f}. This would correspond to choosing an arbitrary function $k_0 = k_0(N_{\mathrm{f}})$ for the global scale occurring in the scaling relations (3.4) and (3.7). Of course, such a function induced by some ad-hoc scale fixing procedure could obscure our scaling relations. For instance, an extreme choice would be given by measuring all dimensionful scales in units of the critical temperature T_{cr}. In this case, the shape of the phase boundary would be a horizontal line at $T/T_{\mathrm{cr}} = 1$ terminating at $N_{\mathrm{f}} = N_{\mathrm{f}}^{\mathrm{cr}}$. Nevertheless, the scaling relations could still be verified, as they would translate into scaling relations for other external scales: e.g., the scale k at which the running coupling acquires a specific value (say $\alpha = 0.322$) would diverge with $N_{\mathrm{f}} \to N_{\mathrm{f}}^{\mathrm{cr}}$ according to $k \sim T_{\mathrm{cr}} |N_{\mathrm{f}} - N_{\mathrm{f}}^{\mathrm{cr}}|^{\frac{1}{\Theta}}$ for fixed T_{cr} (and $\Theta < 0$). This point of view can constitute a different way of verifying our scaling

relations on the lattice. In summary, these considerations demonstrate that the existence of scaling relations has a universal meaning, even though their concrete manifestation can depend on the details of the N_{f}-dependent scale fixing. In particular, the non-analytic structure governed by the exponent Θ always remains.

5. Scaling in low-energy models

Let us study explicitly the scaling of two model observables, namely the quark condensate $\langle \bar{q}q \rangle$ and the (constituent) quark mass M_q, by means of a simple ansatz for the effective potential U for the low-energy sector of QCD:

$$U(\phi) = \frac{1}{2}\mathrm{Tr}\ln\left(\partial^2 + M_\sigma^2(\phi^2)\right) + \frac{N_f^2 - 1}{2}\mathrm{Tr}\ln\left(\partial^2 + M_\pi^2(\phi^2)\right) - N_f N_c \mathrm{Tr}\ln\left(\partial\!\!\!/ + M_q(\phi)\right) . \quad (5.1)$$

Here, ϕ represents a bi-fermionic scalar mean-field, the expectation value of which is related to the chiral condensate. Excitations on top of this condensate correspond to the Goldstone bosons, say the pions, and the sigma meson (radial mode). Mean-field expressions of this type generically arise in many low-energy QCD models such as NJL-type models or the quark-meson model. The Tr ln terms simply correspond to the fluctuation contributions of the chiral mesonic and quark degrees of freedom. The masses M_π and M_σ of these mesons depend on $\partial U/\partial\phi^2$ and a linear combination of $\partial U/\partial\phi^2$ and $\partial^2 U/\partial\phi^2\partial\phi^2$, respectively, see e. g. Refs. [46, 47, 48]. The (constituent) quark mass M_q is given by the product of the Yukawa coupling h and ϕ.

The expression Eq. (5.1) for the effective potential is UV divergent and needs to be regularized in some scheme (belonging to the definition of the model) at an effective regulator scale[5] Λ_{eff}. The IR observables can then be computed by solving Eq. (5.1) self-consistently for $U(\phi)$ together with an initial renormalization condition provided at the scale Λ_{eff}, e.g., $U(\phi)|_{\Lambda_{\mathrm{eff}}} = \frac{1}{2} m_{\Lambda_{\mathrm{eff}}}^2 \phi^2 + \ldots$. The bosonic mass parameter is naturally related to the effective scale, $m_{\Lambda_{\mathrm{eff}}} \sim \Lambda_{\mathrm{eff}}$.

For simplicity, let us consider the model in the limit of large number of colors, $N_c \to \infty$, for a fixed number of flavors N_f by way of example. In this limit only the quark loop survives and we are left with the following equation for the effective potential U:

$$U(\phi) = -N_f N_c \mathrm{Tr}\ln\left(\partial\!\!\!/ + M_q(\phi)\right) . \quad (5.2)$$

Note that the Yukawa coupling is constant in this limit and can therefore be absorbed in a redefinition of the field ϕ, i.e., $M_q = \phi$. The resulting equation for the effective potential can now be solved easily. The pion decay constant is given by the value of ϕ which minimizes the potential U:

$$\left.\frac{\partial U}{\partial \phi}\right|_{\phi=\phi_0} = 0. \quad (5.3)$$

[5]Additionally, IR divergences can occur in the broken phase in the chiral limit, as the Goldstone bosons are massless. Such IR divergencies can be tamed, e.g., by the functional RG, yielding well-defined IR predictions.

We find

$$\phi_0 \simeq \sqrt{N_f N_c}\Lambda_{\rm eff} = \sqrt{N_f N_c} k_0 |N_{\rm f} - N_{\rm f}^{\rm cr}|^{-\frac{1}{\Theta}}, \tag{5.4}$$

where the last step holds near the conformal window, using the relation (3.5). Since the (constituent) quark mass is given by ϕ_0, we conclude that M_q has a scaling behavior near $N_f^{\rm cr}$ identical to that of the critical temperature. We would like to stress that the prefactor $\sqrt{N_f N_c}$ is an outcome of our large-N_c analysis of the low-energy sector. In general, we expect that any observable $\mathcal{O}$ comes along with a complicated pre-factor function $f_{\mathcal{O}}$ depending on the number of flavors N_f and N_c. The determination of this function, e. g. for the constituent mass, may become complicated, depending on the truncations made in the low-energy sector. However, we stress that the N_f dependence coming from the prefactor is subleading compared to the scaling with $|N_{\rm f} - N_{\rm f}^{\rm cr}|$ according to the IR critical exponent $|\Theta|$ of the running coupling.

The scaling of the quark condensate can then be obtained by employing the following relation:

$$|\langle \bar{q} q \rangle| = \frac{m^2_{\Lambda_{\rm eff}}}{h^2}\phi_0 \,. \tag{5.5}$$

Thus we find

$$|\langle \bar{q} q \rangle| \sim k_0^3 |N_{\rm f} - N_{\rm f}^{\rm cr}|^{-\frac{3}{\Theta}} \,. \tag{5.6}$$

The precise proportionality factor will again depend on the number of flavors and colors which, however, cannot modify the scaling behavior with respect to the distance to the conformal window.

In [49], the quark mass, the chiral condensate and the pion decay constant have been computed within truncated Dyson-Schwinger equation for many-flavor QCD. Signatures of the quantum critical point have been identified and the critical exponents have been extracted from a fit to the numerical data available away from the critical point for $N_{\rm f} < N_{\rm f}^{\rm cr}$. In the light of our scaling relation, the results of [49] unfortunately remain somewhat inconclusive as $N_{\rm f}^{\rm cr}$ has been fitted for each IR observable separately (yielding different values). This uncertainty is likely to spoil the fit for the critical exponents. We expect that a more careful analysis in the vicinity of the quantum critical point can easily put our scaling relation to test.

Away from the chiral limit, the current quark mass is expected to modify the scaling relations. A generalized Gell-Mann-Oakes-Renner relation based on the fixed-point scenario in many-flavor QCD has been advocated in [50].

6. Conclusions

Many-flavor deformations of real QCD are a fascinating testing ground for aspects of the chiral structure of QCD-like theories. The existence of a conformal window for larger flavor numbers $N_{\rm f}$ (below the critical value where asymptotic freedom is lost) gives rise to an interesting quantum critical point on the $N_{\rm f}$ axis. Whereas a lot of effort has recently gone

into the determination of the value of $N_{\rm f}$, we have concentrated in this work on the physics in the quantum critical region. Similarly to itinerant fermion systems [51], we expect that the quantum critical point influences the properties of the phase diagram of a number of observables as a function of $N_{\rm f}$ in the neighborhood of the critical point. Note that the quantum phase transition in $N_{\rm f}$ can also be viewed as a higher-dimensional analogue of the Berezinskii-Kosterlitz-Thouless transition [52]. Similar transitions even occur in quantum mechanics which can also be analyzed in an RG language [53].

We have shown in this work, that the scaling of generic IR observables in the chirally broken phase as a function of $N_{\rm f}^{\rm cr} - N_{\rm f}$ exhibits a remarkably universal behavior. Our arguments were based on only few assumptions: the existence of an IR Caswell-Banks-Zaks fixed point in the running of the gauge coupling in the conformal window (which holds by construction if the conformal window exists), and the existence of a critical value of the gauge coupling for triggering chiral symmetry breaking. Whereas the first assumption is a universal statement, the latter assumption needs to be fulfilled only in specific RG schemes and for certain definitions of the coupling. In other schemes and coupling definitions, this assumption may translate into an assumption, for instance, on a sufficiently strong critical behavior of the quark-gluon vertex.

We find that generic IR observables in the χSB phase scale with the distance to the conformal window in a characteristic fashion which is governed by only one independent critical exponent. This critical exponent is directly related to the corresponding critical exponent of the gauge coupling at the Caswell-Banks-Zaks fixed point. This relates universal quantities with each other: chiral IR observables and the IR critical exponent. It establishes a quantitative connection between the chiral structure ($T_{\rm cr}$, $\langle\bar{\psi}\psi\rangle$, f_π, etc.) and the IR gauge dynamics quantified by the critical exponent Θ. Most importantly, it is a parameter free prediction following essentially from scaling arguments.

Recently, the interest in the conformal window of many-flavor QCD has been revived by technicolor scenarios [54, 55] for the Higgs sector of the standard model which have attracted renewed attention [56]. As these scenarios are largely motivated by the hierarchy problem of the standard model, it is important for some so-called walking models that the technicolor sector remains in the vicinity of the quantum critical point over a wide range of scales. The scaling laws should therefore directly apply to such models, even though each model, of course, has a fixed $N_{\rm f}$.

It should be stressed that varying $N_{\rm f}$ as a control parameter for a quantum phase transition corresponds to comparing different theories with each other. Such a comparison is not unique for non-conformal theories but requires a specific choice of a dimensionful scale which is used as one and the same ruler for different theories. In the present work, we have argued that choosing a mid-momentum scale such as the τ mass has many advantages. For small $N_{\rm f}$, this choice has lead us to a simple estimate for the $N_{\rm f}$ dependence of the critical temperature, which is in very good agreement with lattice simulations. This type of argument has already been successfully applied to the PNJL/PQM model, where implementing the linear $N_{\rm f}$ scaling for small $N_{\rm f}$ has helped adjusting the physical parameters, leading to a significant improvement of the thermodynamics properties of the model [37].

Coming back to the scaling relation near the conformal window, we are aware of the

fact that, for instance, the relation (3.4) is difficult to test by lattice gauge theory: neither the fixed-point scenario in the deep IR nor large flavor numbers in the chiral limit are easily accessible. However, given the conceptual simplicity of the fixed-point scenario in combination with χSB, further lattice studies in this direction are certainly worthwhile.

Acknowledgments

The authors thank J.M. Pawlowski for useful discussions and acknowledge support by the DFG under grants Gi 328/1-4, Gi 328/5-1 (Heisenberg program) and FOR 723.

References

[1] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974).

[2] T. Banks and A. Zaks, Nucl. Phys. B **196**, 189 (1982).

[3] V. A. Miransky and K. Yamawaki, Phys. Rev. D **55**, 5051 (1997) [Erratum-ibid. D **56**, 3768 (1997)] [arXiv:hep-th/9611142].

[4] T. Appelquist, J. Terning and L. C. R. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996) [arXiv:hep-ph/9602385].

[5] T. Appelquist and S. B. Selipsky, Phys. Lett. B **400**, 364 (1997) [arXiv:hep-ph/9702404].

[6] T. Schafer and E. V. Shuryak, Rev. Mod. Phys. **70**, 323 (1998) [arXiv:hep-ph/9610451].

[7] M. Velkovsky and E. V. Shuryak, Phys. Lett. B **437**, 398 (1998) [arXiv:hep-ph/9703345].

[8] T. Appelquist, A. Ratnaweera, J. Terning and L. C. R. Wijewardhana, Phys. Rev. D **58**, 105017 (1998) [arXiv:hep-ph/9806472].

[9] F. Sannino and J. Schechter, Phys. Rev. D **60**, 056004 (1999) [arXiv:hep-ph/9903359].

[10] M. Harada and K. Yamawaki, Phys. Rev. Lett. **86** (2001) 757 [hep-ph/0010207].

[11] M. Harada, M. Kurachi and K. Yamawaki, Phys. Rev. D **68**, 076001 (2003) [arXiv:hep-ph/0305018].

[12] H. Gies and J. Jaeckel, Eur. Phys. J. C **46**, 433 (2006) [arXiv:hep-ph/0507171].

[13] J. Braun and H. Gies, Phys. Lett. B **645**, 53 (2007) [arXiv:hep-ph/0512085].

[14] J. Braun and H. Gies, JHEP **0606**, 024 (2006) [arXiv:hep-ph/0602226].

[15] E. Poppitz and M. Unsal, JHEP **0909**, 050 (2009) [arXiv:0906.5156 [hep-th]]; JHEP **0912**, 011 (2009) [arXiv:0910.1245 [hep-th]].

[16] A. Armoni, arXiv:0907.4091 [hep-ph].

[17] F. Sannino, Phys. Rev. D **80**, 065011 (2009) [arXiv:0907.1364 [hep-th]]; Nucl. Phys. B **830**, 179 (2010) [arXiv:0909.4584 [hep-th]]; arXiv:0911.0931 [hep-ph].

[18] J. B. Kogut, M. Stone, H. W. Wyld, J. Shigemitsu, S. H. Shenker and D. K. Sinclair, Phys. Rev. Lett. **48**, 1140 (1982).

[19] R. V. Gavai, Nucl. Phys. B **269**, 530 (1986).

[20] M. Fukugita, S. Ohta and A. Ukawa, Phys. Rev. Lett. **60**, 178 (1988).

[21] F. R. Brown, H. Chen, N. H. Christ, Z. Dong, R. D. Mawhinney, W. Schaffer and A. Vaccarino, Phys. Rev. D **46**, 5655 (1992) [arXiv:hep-lat/9206001].

[22] P. H. Damgaard, U. M. Heller, A. Krasnitz and P. Olesen, Phys. Lett. B **400**, 169 (1997) [arXiv:hep-lat/9701008].

[23] Y. Iwasaki, K. Kanaya, S. Kaya, S. Sakai and T. Yoshie, Phys. Rev. D **69**, 014507 (2004) [arXiv:hep-lat/0309159].

[24] S. Catterall and F. Sannino, Phys. Rev. D **76**, 034504 (2007) [arXiv:0705.1664 [hep-lat]].

[25] T. Appelquist, G. T. Fleming and E. T. Neil, Phys. Rev. Lett. **100**, 171607 (2008) [Erratum-ibid. **102**, 149902 (2009)] [arXiv:0712.0609 [hep-ph]].

[26] A. Deuzeman, M. P. Lombardo and E. Pallante, Phys. Lett. B **670**, 41 (2008) [arXiv:0804.2905 [hep-lat]]; arXiv:0810.3117 [hep-lat]; PoS **LATTICE2008**, 060 (2008) [arXiv:0810.1719 [hep-lat]].

[27] A. Deuzeman, M. P. Lombardo and E. Pallante, arXiv:0904.4662 [hep-ph].

[28] T. Appelquist, G. T. Fleming and E. T. Neil, Phys. Rev. D **79**, 076010 (2009) [arXiv:0901.3766 [hep-ph]].

[29] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, arXiv:0907.4562 [hep-lat].

[30] T. DeGrand, Phys. Rev. D **80**, 114507 (2009) [arXiv:0910.3072 [hep-lat]].

[31] Z. Fodor, K. Holland, J. Kuti, D. Nogradi and C. Schroeder, arXiv:0911.2463 [hep-lat].

[32] E. Pallante, arXiv:0912.5188 [hep-lat].

[33] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).

[34] R. S. Chivukula, Phys. Rev. D **55**, 5238 (1997) [arXiv:hep-ph/9612267].

[35] J. Braun, arXiv:0908.1543 [hep-ph].

[36] F. Karsch, E. Laermann and A. Peikert, Nucl. Phys. B **605**, 579 (2001) [arXiv:hep-lat/0012023].

[37] B. J. Schaefer, J. M. Pawlowski and J. Wambach, Phys. Rev. D **76**, 074023 (2007) [arXiv:0704.3234 [hep-ph]]; B. J. Schaefer, M. Wagner and J. Wambach, arXiv:0910.5628 [hep-ph].

[38] R. Alkofer and L. von Smekal, Phys. Rept. **353**, 281 (2001) [arXiv:hep-ph/0007355]; C. S. Fischer and R. Alkofer, Phys. Lett. B **536**, 177 (2002) [arXiv:hep-ph/0202202]; C. S. Fischer, J. Phys. G **32**, R253 (2006) [arXiv:hep-ph/0605173].

[39] C. Wetterich, Phys. Lett. B **301**, 90 (1993).

[40] M. Reuter, arXiv:hep-th/9602012; D. F. Litim and J. M. Pawlowski, arXiv:hep-th/9901063; H. Gies, arXiv:hep-ph/0611146; J. M. Pawlowski, Annals Phys. **322**, 2831 (2007) [arXiv:hep-th/0512261].

[41] H. Gies, J. Jaeckel and C. Wetterich, Phys. Rev. D **69**, 105008 (2004) [arXiv:hep-ph/0312034].

[42] M. Reuter and C. Wetterich, Phys. Rev. D **56**, 7893 (1997) [arXiv:hep-th/9708051].

[43] H. Gies, Phys. Rev. D **66**, 025006 (2002) [arXiv:hep-th/0202207].

[44] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B **400**, 379 (1997) [arXiv:hep-ph/9701390]; M. Czakon, Nucl. Phys. B **710**, 485 (2005) [arXiv:hep-ph/0411261].

[45] T. A. Ryttov and F. Sannino, Phys. Rev. D **78**, 065001 (2008) [arXiv:0711.3745 [hep-th]]; F. Sannino, Phys. Rev. D **79**, 096007 (2009) [arXiv:0902.3494 [hep-ph]].

[46] D. U. Jungnickel and C. Wetterich, Phys. Rev. D **53**, 5142 (1996) [arXiv:hep-ph/9505267].

[47] B. J. Schaefer and H. J. Pirner, Nucl. Phys. A **660**, 439 (1999) [arXiv:nucl-th/9903003].

[48] J. Braun, B. Klein and H. J. Pirner, Phys. Rev. D **71**, 014032 (2005) [arXiv:hep-ph/0408116].

[49] O. Gromenko, arXiv:0710.1591 [hep-ph].

[50] F. Sannino, Phys. Rev. D **80**, 017901 (2009) [arXiv:0811.0616 [hep-ph]].

[51] P. Jakubczyk, P. Strack, A. A. Katanin and W. Metzner, Phys. Rev. B **77**, 195120 (2008) [arXiv:0802.1868 [cond-mat.str-el]].

[52] D. B. Kaplan, J. W. Lee, D. T. Son and M. A. Stephanov, Phys. Rev. D **80**, 125005 (2009) [arXiv:0905.4752 [hep-th]].

[53] S. Moroz and R. Schmidt, arXiv:0909.3477 [hep-th].

[54] S. Weinberg, Phys. Rev. D **19**, 1277 (1979); L. Susskind, Phys. Rev. D **20**, 2619 (1979); E. Farhi and L. Susskind, Phys. Rept. **74**, 277 (1981).

[55] B. Holdom, Phys. Rev. D **24**, 1441 (1981); K. Yamawaki, M. Bando and K. i. Matumoto, Phys. Rev. Lett. **56**, 1335 (1986); T. W. Appelquist, D. Karabali and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); E. Eichten and K. D. Lane, Phys. Lett. B **90**, 125 (1980).

[56] D. K. Hong, S. D. H. Hsu and F. Sannino, Phys. Lett. B **597**, 89 (2004) [arXiv:hep-ph/0406200]; F. Sannino and K. Tuominen, Phys. Rev. D **71**, 051901 (2005) [arXiv:hep-ph/0405209]; D. D. Dietrich, F. Sannino and K. Tuominen, Phys. Rev. D **72**, 055001 (2005) [arXiv:hep-ph/0505059]; D. D. Dietrich and F. Sannino, Phys. Rev. D **75**, 085018 (2007) [arXiv:hep-ph/0611341]; T. A. Ryttov and F. Sannino, Phys. Rev. D **76**, 105004 (2007) [arXiv:0707.3166 [hep-th]]; O. Antipin and K. Tuominen, arXiv:0909.4879 [hep-ph].